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Optimal embeddings of generalized ladders into hypercubes

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Dedicated to the memory of Ivan Havel

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0. Introduction

Embeddings of graphs from one class of graphs into graphs from another class have an important application in computer science. Any finite graph can be considered as a model of a parallel computer — vertices correspond to processors and edges model communication lines between them. Consider a simple type of simulation between parallel computers where any processor of a simulated computer is replaced by a processor of a target computer and communication lines correspond to communication lines. Then there exists a one-to-one correspondence between embeddings and these simulations. Thus a rapid development of computer technology and its theoretical foundations requires the study of embeddings between special classes of graphs. Principles and details of this view can be found in the survey paper of Monien and Sudborough [9] and in the monograph by Leighton [8]. Computer science prefers classes of graphs that model feasible and technically constructible computers, and one such class of graphs is the class of all hypercubes. The second consequence of this view is an effort to optimize parameters of constructed embeddings. These facts motivate investigation of embeddings of a graph into the smallest possible hypercube. To formalize this requirement, we say that an embedding of a graph into a hypercube is *optimal* if an expansion of the embedding, that is, the ratio of the number of vertices of the hypercube to the number of vertices of the original graph is less than two.

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Numerous papers have been devoted to a study of embeddings and/or optimal embeddings of some graphs into hypercubes, see [6,8,9]. The pioneering paper is probably due to Nebeský [10]. In early 1970s, Havel and Liebl had built on this paper, and Havel had formulated a conjecture that still remains open:

Havel's Conjecture. Every balanced binary tree has an optimal embedding into a hypercube.

On the other hand, Corneil and Wagner [11] proved that to decide whether there exists an embedding of a given tree into a given hypercube is a NP-complete problem. Havel and Liebl [7] have begun to investigate embeddings of caterpillars (i.e. a special type of binary trees) into hypercubes. A survey paper on embeddings of caterpillars into hypercubes is due to Harary (see [6]). Most recent papers concerning optimal embeddings of caterpillars in hypercubes are due to Dvořák et al. [5] and to the authors of this note [3].

The effort to find other classes of caterpillars with optimal embeddings into hypercubes has motivated investigations of embeddings of ladders (defined below) into hypercubes. First, Bezrukov et al. (see [1,2], modified version of [1]) proved that any ladder with even rungs has an optimal embedding. In [4], the authors of this note showed that there exists a ladder with odd rungs that does not have an optimal embedding into a hypercube, while on the other hand, any ladder with odd rungs greater than 6 does have an optimal embedding into a hypercube. The aim of this paper is to generalize both results. We define a new class of graphs — the so-called generalized ladders — that includes all ladders. We prove that any balanced bipartite generalized ladder with even rungs (this class contains all ladders with even rungs) has an optimal embedding into a hypercube (see Corollary 2.2). Thus we can say that Corollary 2.2 gives the affirmative solution of Havel conjecture for generalized ladders with even rungs and generalized cyclic ladders with even rungs (in both cases they are not trees). The method of the proof is a generalization of that in [2]. We also prove that there exists a subclass of all balanced bipartite generalized ladders with odd rungs — the so-called pretty generalized ladders — such that any pretty generalized ladder has an optimal embedding into a hypercube (see Theorem 3.4). Any balanced ladder with odd rungs greater than 6 is a pretty generalized ladder. Our second result generalizes some results in [4] (Theorem 5.1) and its proof exploits the method in [4] (originally of [1]). Unfortunately we did not succeed in unifying the proofs of the two results.

1. Hypercubes and ladders

Let $\mathbb{N} = \{0, 1, \ldots\}$ denote the set of all natural numbers and $\equiv$ denote the equivalence on $\mathbb{N}$ such that $n \equiv m$ just when $n \equiv m \bmod 2$ (i.e., n and m have the same parity). Since we use only the equivalence by modulo 2 we write only $\equiv$ instead of usual $\equiv_2$.

Any set mentioned in this note other than the set $\mathbb{N}$ will be *finite*. For a set X, let $|X|$ denote its size and $\exp X$ denote the set of all subsets of X. For any two sets $A, B \subseteq X$, write $\Delta(A,B) = (A \setminus B) \cup (B \setminus A)$. It is well known that the operation Δ determines an Abelian group with the neutral element $\emptyset$ on the set $\exp X$ such that each its element is of order 2. For $\mathscr{A} \subseteq \exp X$, set $\max(\mathscr{A}) = \bigcup_{A \in \mathscr{A}} A$ and $\min(\mathscr{A}) = \bigcap_{A \in \mathscr{A}} A$. If $B \subseteq X$ then denote $\Delta(\mathscr{A}, B) = \{\Delta(A,B) \mid A \in \mathscr{A}\}$. For $x \in X$ we shall write $\Delta(A,x)$ or $\Delta(\mathscr{A},x)$ instead of $\Delta(A,\{x\})$ of $\Delta(\mathscr{A},\{x\})$.

A *graph* $\boldsymbol{G}$ is a pair (V,E) where V is a set (we call V the *set of vertices* of $\boldsymbol{G}$ and write $V(\boldsymbol{G}) = V$) and E is a set of two-element subsets of V (we call E the *set of edges* of $\boldsymbol{G}$ and write $E(\boldsymbol{G}) = E$). A one-to-one sequence $\boldsymbol{P} = (x_0, x_1, \dots, x_n)$ of vertices of $\boldsymbol{G}$ is called a *path* between x_0 and x_n in a graph $\boldsymbol{G}$ if $\{x_i, x_{i+1}\} \in E(\boldsymbol{G})$ for all $i = 0, 1, \dots, n-1$. We say that the *length* of $\boldsymbol{P}$ is n. Then $V(\boldsymbol{P}) = \{x_i \mid i = 0, 1, \dots, n\}$ and $E(\boldsymbol{P}) = \{\{x_i, x_{i+1}\} \mid i = 0, 1, \dots, n-1\}$. If we add to $\boldsymbol{P}$ an edge $\{x_0, x_n\} \in E(\boldsymbol{G})$, then $\boldsymbol{P}$ is called a *cycle* of $\boldsymbol{G}$ of length $n+1$ (if $\boldsymbol{P}$ is a cycle then $E(\boldsymbol{P}) = \{\{x_i, x_{i+1}\} \mid i = 0, 1, \dots, n-1\} \cup \{\{x_0, x_n\}\}$).

Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be graphs. A mapping $f : V(\boldsymbol{G}) \to V(\boldsymbol{H})$ such that $\{f(x), f(y)\} \in E(\boldsymbol{H})$ for every $\{x, y\} \in E(\boldsymbol{G})$ is called a *graph homomorphism* from $\boldsymbol{G}$ to $\boldsymbol{H}$. A one-to-one graph homomorphism f from $\boldsymbol{G}$ to $\boldsymbol{H}$ is called an *embedding* from $\boldsymbol{G}$ into $\boldsymbol{H}$, and if $\operatorname{Im}(f) = \{f(x) \mid x \in V(\boldsymbol{G})\} = \mathscr{A}$ then we say that f is an *$\mathscr{A}$-embedding*.

We recall that a graph $\boldsymbol{G}$ is called *connected* if there exists a path between each pair of distinct vertices of $V(\boldsymbol{G})$ and is called *bipartite* if there exists a mapping $f : V(\boldsymbol{G}) \to \{0,1\}$ such that $f(x) \neq f(y)$ for all $\{x, y\} \in E(\boldsymbol{G})$; if, moreover $|f^{-1}(0)| = |f^{-1}(1)|$, we say that $\boldsymbol{G}$ is *balanced*. Observe that for any connected bipartite graph $\boldsymbol{G}$, there exist exactly two mappings with $f(x) \neq f(y)$ for all $\{x, y\} \in E(\boldsymbol{G})$. We recall that a graph $\boldsymbol{G}$ is bipartite if and only if any cycle of $\boldsymbol{G}$ has an even length.

A graph $\boldsymbol{G}$ is called a *generalized ladder* if there are disjoint paths $\boldsymbol{P}_0 = (x_0, x_1, \dots, x_n)$ and $\boldsymbol{P}_1 = (y_0, y_1, \dots, y_m)$ of $\boldsymbol{G}$ (they are called *leading paths* of $\boldsymbol{G}$) and p disjoint paths $\boldsymbol{R}_i = (z_{i,0}, z_{i,1}, \dots, z_{i,\ell_i})$ for $i = 0, 1, \dots, p-1$ (they called *rungs* of $\boldsymbol{G}$) such that

(a) $V(\boldsymbol{P}_0) \cap V(\boldsymbol{R}_i) = \{z_{i,0} = x_{j_i}\}$ for some $j_i \in \{0, 1 \dots, n\}$, $V(\boldsymbol{P}_1) \cap V(\boldsymbol{R}_i) = \{z_{i,\ell_i} = y_{k_i}\}$ for some $k_i \in \{0, 1, \dots, m\}$ for all $i = 0, 1, \dots, p-1$, and $j_i < j_{i'}$ and $k_i < k_{i'}$ for all $i, i' = 0, 1, \dots, p-1$ with $i < i'$;

(b)

$$V(\boldsymbol{G}) = \left(\bigcup_{i=0}^{p-1} V(\boldsymbol{R}_i) \right) \cup V(\boldsymbol{P}_0) \cup V(\boldsymbol{P}_1) \quad \text{and}$$

$$E(\boldsymbol{G}) = \left(\bigcup_{i=0}^{p-1} E(\boldsymbol{R}_i) \right) \cup E(\boldsymbol{P}_0) \cup E(\boldsymbol{P}_1).$$

The sequences $(j_0, j_1, \dots, j_{p-1})$ and $(k_0, k_1, \dots, k_{p-1})$ are called *determining sequences* of $\boldsymbol{G}$. We say that $\boldsymbol{G}$ has *even rungs* (or *odd rungs*) whenever ℓ_i is odd (or even) (i.e., if the number of vertices of $V(\boldsymbol{R}_i)$ is even (or odd, respectively)) for all $i = 0, 1, \dots, p-1$.

We shall construct our embeddings by induction. The induction exploits special subgraphs of generalized ladders that we now define. For any pair (q,r) of natural numbers with $q\leqslant n$, $r\leqslant m$ define $\mu(\boldsymbol{G},q,r)=\max\{i\in\mathbb{N}\mid j_i\leqslant q \text{ and } k_i\leqslant r\}$, $\tau(\boldsymbol{G},q,r)=\min(i\in\mathbb{N}\mid j_i\geqslant r \text{ and } k_i\geqslant q\}$. For any pairs (q,r) and (s,t) of natural numbers, let ${}_{q,r}\boldsymbol{G}$, $\boldsymbol{G}_{q,r}$ and ${}_{q,r}\boldsymbol{G}_{s,t}$ denote induced subgraphs of $\boldsymbol{G}$ on subsets:

$$V({}_{q,r}\boldsymbol{G})=\{x_i\mid 0\leqslant i\leqslant q\}\cup\{y_i\mid 0\leqslant i\leqslant r\}\cup\left(\bigcup_{i=0}^{\mu(\boldsymbol{G},q,r)}V(\boldsymbol{R}_i)\right),$$

$$V(\boldsymbol{G}_{q,r})=\{x_i\mid q\leqslant i\leqslant n\}\cup\{y_i\mid r\leqslant i\leqslant m\}\cup\left(\bigcup_{i=\tau(\boldsymbol{G},q,r)}^{p-1}V(\boldsymbol{R}_i)\right)$$

and

$$V({}_{q,r}\boldsymbol{G}_{s,t})=\{x_i\mid s\leqslant i\leqslant q\}\cup\{y_i\mid t\leqslant i\leqslant r\}\cup\left(\bigcup_{i=\tau(\boldsymbol{G},s,t)}^{\mu(\boldsymbol{G},q,r)}V(\boldsymbol{R}_i)\right).$$

Clearly, for any (s,t) and (q,r) with $s\leqslant q$ and $t\leqslant r$, the graphs ${}_{q,r}\boldsymbol{G}$, $\boldsymbol{G}_{q,r}$ and ${}_{q,r}\boldsymbol{G}_{s,t}$ are generalized ladders such that leading paths of ${}_{q,r}\boldsymbol{G}$ are $(x_0,x_1,\ldots,x_q)$ and $(y_0,y_1,\ldots,y_r)$, determining sequences of ${}_{q,r}\boldsymbol{G}$ are sequences $(j_0,j_1,\ldots,j_{\mu(\boldsymbol{G},q,r)})$ and $(k_0,k_1,\ldots,k_{\mu(\boldsymbol{G},q,r)})$ and rungs of ${}_{q,r}\boldsymbol{G}$ are $\boldsymbol{R}_0,\boldsymbol{R}_1,\ldots,\boldsymbol{R}_{\mu(\boldsymbol{G},q,r)}$, leading paths of $\boldsymbol{G}_{q,r}$ are $(x_q,x_{q+1},\ldots,x_n)$ and $(y_r,y_{r+1},\ldots,y_m)$, determining sequences of $\boldsymbol{G}_{q,r}$ are $(j_{\tau(\boldsymbol{G},q,r)},\ j_{\tau(\boldsymbol{G},q,r)+1},\ldots,j_{p-1})$ and $(k_{\tau(\boldsymbol{G},q,r)},k_{\tau(\boldsymbol{G},q,r)+1},\ldots,k_{p-1})$ and rungs of $\boldsymbol{G}_{q,r}$ are $\boldsymbol{R}_{\tau(\boldsymbol{G},q,r)}$, $\boldsymbol{R}_{\tau(\boldsymbol{G},q,r)+1},\ldots,\boldsymbol{R}_{p-1}$, and leading paths of ${}_{q,r}\boldsymbol{G}_{s,t}$ are $(x_s,x_{s+1},\ldots,x_q)$ and $(y_t,y_{t+1},\ldots,y_r)$, determining sequences of ${}_{q,r}\boldsymbol{G}_{s,t}$ are $(j_{\tau(\boldsymbol{G},s,t)},\ j_{\tau(\boldsymbol{G},s,t)+1},\ldots,j_{\mu(\boldsymbol{G},q,r)})$ and $(k_{\tau(\boldsymbol{G},s,t)},k_{\tau(\boldsymbol{G},s,t)+1},\ldots,k_{\mu(\boldsymbol{G},q,r)})$ and rungs of ${}_{q,r}\boldsymbol{G}_{s,t}$ are $\boldsymbol{R}_{\tau(\boldsymbol{G},s,t)}$, $\boldsymbol{R}_{\tau(\boldsymbol{G},s,t)+1},\ldots,\boldsymbol{R}_{\mu(\boldsymbol{G},q,r)}$.

A graph $\boldsymbol{H}$ is called a *generalized cyclic ladder* if it is obtained from a generalized ladder $\boldsymbol{G}$ by adding edges $\{x_0,x_n\}$ and $\{y_0,y_m\}$. Thus $V(\boldsymbol{H})=V(\boldsymbol{G})$ and $E(\boldsymbol{H})=E(\boldsymbol{G})\cup\{\{x_0,x_n\},\{y_0,y_m\}\}$, and we shall write $\boldsymbol{H}=\mathrm{cyclic}(\boldsymbol{G})$. Observe that there exist non-isomorphic generalized ladders $\boldsymbol{G}_0$ and $\boldsymbol{G}_1$ with $\mathrm{cyclic}(\boldsymbol{G}_0)=\mathrm{cyclic}(\boldsymbol{G}_1)$. We say that a generalized cyclic ladder $\boldsymbol{H}$ has *even rungs* (or *odd rungs*) if there exists a generalized ladder $\boldsymbol{G}$ with even rungs (or odd rungs) such that $\boldsymbol{H}=\mathrm{cyclic}(\boldsymbol{G})$ (it easy to see that these definitions are correct).

We recall that a graph $\boldsymbol{G}$ is called a *ladder* (or a *cyclic ladder*) if $\boldsymbol{G}$ is a generalized ladder (or a generalized cyclic ladder) such that $n=m$ and determining sequences of $\boldsymbol{G}$ coincide and have a form $(0,1,\ldots,n)$.

The following statement is a combination of well-known and/or obvious facts.

Statement 1.1. *Let $\boldsymbol{G}$ be a generalized ladder.*

(1) *If $\boldsymbol{G}$ has odd rungs or even rungs, then $\boldsymbol{G}$ is a bipartite if and only if either $j_i\equiv k_i$ for all $i=0,1,\ldots,p-1$ or $j_i\not\equiv k_i$ for all $i=0,1,\ldots,p-1$.*
(2) *If $\boldsymbol{G}$ has even rungs and $p>0$, then $\boldsymbol{G}$ is balanced bipartite if and only if either $n\equiv m$ and $j_i\equiv k_i$ for all $i=0,1,\ldots,p-1$, or both n and m are odd and $j_i\not\equiv k_i$ for all $i=0,1,\ldots,p-1$.*

(3) *If* $\boldsymbol{G}$ *has odd rungs,* $p > 0$, $j_i \equiv k_i$ *for all* $i = 0, 1, \ldots, p-1$, *and* $j_{2i} \not\equiv j_{2i+1}$ *for all* $i = 0, 1, \ldots, \lfloor (p-2)/2 \rfloor$, *then* $\boldsymbol{G}$ *is a balanced bipartite graph if and only if either* n, m *and* $p-1$ *are odd, or* $p-1$ *and* j_{p-1} *are even* $n \not\equiv m$.

(4) *If* $\boldsymbol{G}$ *has odd rungs,* $p > 0$, $j_i \not\equiv k_i$ *for all* $i = 0, 1, \ldots, p-1$, *and* $j_{2i} \not\equiv j_{2i+1}$ *for all* $i = 0, 1, \ldots, \lfloor (p-2)/2 \rfloor$, *then* $\boldsymbol{G}$ *is a balanced bipartite graph if and only if either* p *is even and* $n \equiv m$, *or both* j_{p-1} *and* n *are even and both* p *and* m *are odd, or* m *is even and* p, j_{p-1}, n *are odd.*

(5) *If* $\boldsymbol{G}$ *has even rungs,* $n \equiv m$ *and* $j_i \equiv k_i$ *for all* $i = 0, 1, \ldots, p-1$, *then* ${}_{q,r}\boldsymbol{G}$ *and* $\boldsymbol{G}_{q,r}$ *are balanced bipartite graphs for any pair* (q, r) *with* $q \leqslant n$, $r \leqslant m$ *and* $q \equiv r$.

(6) *If* $q, r \in \mathbb{N}$ *with* $q \leqslant n$, $r \leqslant m$, *then* $V({}_{q,r}\boldsymbol{G}) \cap V(\boldsymbol{G}_{q+1,r+1}) = \emptyset$. *Further,* $V(\boldsymbol{G}) = V({}_{q,r}\boldsymbol{G}) \cup V(\boldsymbol{G}_{q+1,r+1})$ *and* $E(\boldsymbol{G}) = E({}_{q,r}\boldsymbol{G}) \cup E(\boldsymbol{G}_{q+1,r+1}) \cup \{\{x_q, x_{q+1}\}, \{y_r, y_{r+1}\}\}$ *if and only if* $\mu(\boldsymbol{G}, q, r) + 1 = \tau(\boldsymbol{G}, q+1, r+1)$.

We say that a generalized ladder $\boldsymbol{G}$ is *nice* if $\boldsymbol{G}$ has even rungs, $n \equiv m$ and $j_i \equiv k_i$ for all $i = 0, 1, \ldots, p-1$. By straightforward calculation, we obtain the following statement.

Statement 1.2. *Let* $\boldsymbol{G}$ *be a generalized ladder and* $\boldsymbol{H} = \mathrm{cyclic}(\boldsymbol{G})$.

(1) *If* $\boldsymbol{G}$ *has even or odd rungs, then* $\boldsymbol{H}$ *is bipartite if and only if* n *and* m *are odd and either* $j_i \equiv k_i$ *for all* $i = 0, 1, \ldots, p-1$ *or* $j_i \not\equiv k_i$ *for all* $i = 0, 1, \ldots, p-1$.

(2) *If* $\boldsymbol{G}$ *has even rungs and* $\boldsymbol{H}$ *is bipartite then* $\boldsymbol{H}$ *is balanced.*

(3) *If* $\boldsymbol{G}$ *has odd rungs and* $\boldsymbol{H}$ *is bipartite then* $\boldsymbol{H}$ *is balanced if and only if* p *is even.*

(4) *If* $\boldsymbol{G}$ *has even rungs, both* n *and* m *are odd and* $j_i \not\equiv k_i$ *for all* $l = 0, 1, \ldots, p-1$, *then there exists a nice generalized ladder* $\boldsymbol{G}_1$ *with* $\boldsymbol{H} = \mathrm{cyclic}(\boldsymbol{G}_1)$. *Lengths of leading paths of* $\boldsymbol{G}_1$ *are odd.*

These two statements form a basis of an induction used to construct our embeddings.

For a set X, let $\boldsymbol{Q}_X$ be a graph such that $V(\boldsymbol{Q}_X) = \exp X$ and $E(\boldsymbol{Q}_X) = \{\{A, B\} \mid A, B \subseteq X, |\Delta(A, B)| = 1\}$. Then we say that $\boldsymbol{Q}_X$ is a *hypercube* over X. Clearly, a hypercube over any set X is a connected and bipartite balanced graph. If the set X is clear from the context then we shall use $\boldsymbol{Q}$ instead of $\boldsymbol{Q}_X$. An embedding f from a graph $\boldsymbol{G}$ into a hypercube $\boldsymbol{Q}$ over a set X is called *optimal* whenever $2^{|X|-1} < |V(\boldsymbol{G})| \leqslant 2^{|X|}$.

For a set X, a subset $\mathscr{A} \subseteq \exp X$ is a *subcube* of $\boldsymbol{Q}_X$ if there exist A, $B \subseteq X$ with $\mathscr{A} = \{C \subseteq X \mid A \subseteq C \subseteq B\}$. For a subcube $\mathscr{A}$, set $\mathrm{dir}(\mathscr{A}) = B \setminus A$ $(= \max(\mathscr{A}) \setminus \min(\mathscr{A}))$. For $x \in X$ and a subcube $\mathscr{A}$ of $\boldsymbol{Q}$, it is easy to see that if $x \in \mathrm{dir}(\mathscr{A})$ then $\Delta(\mathscr{A}, x) = \mathscr{A}$, and if $x \notin \mathrm{dir}(\mathscr{A})$ then $\Delta(\mathscr{A}, x)$ is a subcube of $\boldsymbol{Q}$ disjoint with $\mathscr{A}$. If $x \notin \mathrm{dir}(\mathscr{A})$ then $\Delta(\mathscr{A}, x)$ is called an x-*brother* of $\mathscr{A}$. We say that a subcube $\mathscr{B}$ is a *brother* of $\mathscr{A}$ if it is an x-brother for some $x \in X$. Generally, subcubes $\mathscr{A}$ and $\mathscr{B}$ are x-*neighbouring* for $x \in X$ if $\mathscr{A}$ is a subset of the x-brother of $\mathscr{B}$ or $\mathscr{B}$ is a subset of the x-brother of $\mathscr{A}$, and analogously we say that $\mathscr{A}$ is *neighbouring* of $\mathscr{B}$ if $\mathscr{A}$ is x-neighbouring of $\mathscr{B}$ for some $x \in X$.

Since for any subcube $\mathscr{A}$ of $\boldsymbol{Q}_X$ the induced subgraph of $\boldsymbol{Q}_X$ on the set $\mathscr{A}$ is isomorphic to the hypercube over dir($\mathscr{A}$), subcubes are natural tool for induction constructions in hypercubes. The disadvantage of using subcubes is the fact that their sizes are only power of 2. This leads to the following generalization of subcubes. The notions of a *dense set*, a *canonical decomposition* of a dense set and a *natural brother* of a dense set are defined by induction.

The initial step. Any subcube $\mathscr{A}$ is a dense set with a canonical decomposition $(\mathscr{A})$ and any brother of $\mathscr{A}$ is a natural brother of $\mathscr{A}$.

The induction step. Let $\mathscr{A}$ be a subcube with a brother $\mathscr{B}$. If $\mathscr{A}'$ is a dense set with $\mathscr{A}' \subseteq \mathscr{B}$ then $\mathscr{A} \cup \mathscr{A}'$ is a dense set. If $\mathscr{A}' \neq \mathscr{B}$ then $(\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_n, \mathscr{A})$ is a canonical decomposition of $\mathscr{A} \cup \mathscr{A}'$ whenever $(\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_n)$ is a canonical decomposition of $\mathscr{A}'$, and $\mathscr{B}'$ is a natural brother of $\mathscr{A} \cup \mathscr{A}'$ whenever $\mathscr{B}'$ is a natural brother of $\mathscr{A}'$.

We recall several properties of dense sets that will be exploited later.

Statement 1.3 (Bezrukov et al. [1] or Caha and Koubek [4]). *Let $\boldsymbol{Q}$ be a hypercube over a set X. Then*

(1) *for every $n \in \mathbb{N}$ with $n \leqslant 2^{|X|}$ there exists a dense set $\mathscr{A}$ of $\boldsymbol{Q}$ with $|\mathscr{A}| = n$*;
(2) *if $\mathscr{B}$ is a subcube of $\boldsymbol{Q}$ then for every dense set $\mathscr{A}$ of $\boldsymbol{Q}$ with $\mathscr{A} \subseteq \mathscr{B}$ the set $\mathscr{B} \setminus \mathscr{A}$ is also dense*;
(3) *if $\mathscr{A}$ is a dense set of $\boldsymbol{Q}$ with a canonical decomposition $(\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_n)$ then $(\mathscr{A}_{i_0}, \mathscr{A}_{i_1}, \ldots, \mathscr{A}_{i_k})$ is a canonical decomposition of a dense set $\bigcup_{j=0}^{k} \mathscr{A}_{i_j}$ for any increasing sequence $0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant n$ of natural numbers*;
(4) *if $\mathscr{A}$ and $\mathscr{B}$ are dense sets of $\boldsymbol{Q}$ such that $\mathscr{B} \subseteq \mathscr{C}$ for some natural brother $\mathscr{C}$ of $\mathscr{A}$, then $\mathscr{A} \cup \mathscr{B}$ is a dense set of $\boldsymbol{Q}$*;
(5) *if $(j_0, j_1, \ldots, j_n)$ is an increasing sequence of positive integers, then for every dense set $\mathscr{A}$ of $\boldsymbol{Q}$ with $|\mathscr{A}| = \sum_{i=0}^{n} 2^{j_i}$ and for every canonical decomposition $(\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_m)$ of $\mathscr{A}$ we have $n = m$ and $|\mathscr{A}_i| = 2^{j_i}$ for all $i = 0, 1, \ldots, n$*;
(6) *if $\mathscr{A}$ is a dense set of $\boldsymbol{Q}$ then for every $m \in \mathbb{N}$ with $|\mathscr{A}| \leqslant m \leqslant 2^{|X|}$ there exist two dense sets $\mathscr{B}$ and $\mathscr{C}$* of $\boldsymbol{Q}$ *(possibly empty) such that*
   (a) *$\mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$ and $\mathscr{A} \cup \mathscr{B}$ are dense sets of $\boldsymbol{Q}$, $|\mathscr{A} \cup \mathscr{B} \cup \mathscr{C}| = m$ and there exists $k \in \mathbb{N}$ (uniquely determined) such that $|\mathscr{A} \cup \mathscr{B}|$ is a multiple of 2^k and $|\mathscr{C}| < 2^k$*;
   (b) *$\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are pairwise disjoint*;
   (c) *there exists a canonical decomposition $\mathfrak{B} = (\mathscr{B}_0, \mathscr{B}_1, \ldots, \mathscr{B}_q)$ of $\mathscr{B}$ such that $\mathscr{B}_i$ is a natural brother of the dense set $\mathscr{A} \cup (\bigcup_{j=i+1}^{q} \mathscr{B}_j)$ for all $i = 0, 1, \ldots, q$*;
   (d) *there exists a natural brother $\mathscr{D}$ of the dense set $\mathscr{A} \cup \mathscr{B}$ with $\mathscr{C} \subsetneq \mathscr{D}$.*

By [4], a canonical decomposition of a dense set is not determined uniquely but, by Statement 1.3(5), the size of the ith member of a canonical decomposition of a dense set is uniquely determined by the size of the dense set.

Since our constructions exploit the combinatorial properties of subcubes and dense sets, we use the subset language to describe hypercubes and their subsets.

2. Generalized ladders with even rungs

The aim of this section is to generalize the result of [2] and to prove that any nice generalized ladder has an optimal embedding. The proof generalizes the method from [2]. Assume that $\boldsymbol{G}$ is a nice generalized ladder with leading paths $\boldsymbol{P}_0=(x_0,x_1,\ldots,x_n)$ and $\boldsymbol{P}_1=(y_0,y_1,\ldots,y_m)$, determining sequences $(j_0,j_1,\ldots,j_{p-1})$, $(k_0,k_1,\ldots,k_{p-1})$ and rungs $\boldsymbol{R}_i=(z_{i,0}=x_{j_i},z_{i,1},\ldots,z_{i,\ell_i}=y_{k_i})$ for all $i=0,1,\ldots,p-1$, and assume that X is a set with $2^{|X|-1}<|V(\boldsymbol{G}|\leqslant 2^{|X|}$. We shall construct an optimal embedding of $\boldsymbol{G}$ into $\boldsymbol{Q}_X$, by induction over a size of X. If there exist $q,r\in\mathbb{N}$ with $q\equiv r$, $\mu(\boldsymbol{G},q,r)+1=\tau(\boldsymbol{G},q+1,r+1)$ and $|V({}_{q,r}\boldsymbol{G})|$, $|V(\boldsymbol{G}_{q+1,r+1})|\leqslant 2^{|X|-1}$, then we choose $x\in X$ and, by the induction hypothesis, there exist two optimal embeddings $g_0:{}_{q,r}\boldsymbol{G}\to\boldsymbol{Q}_{X\setminus\{x\}}$ and $g_1:\boldsymbol{G}_{q+1,r+1}\to\boldsymbol{Q}_{X\setminus\{x\}}$ with $g_0(x_q)=g_1(x_{q+1})$, $g_0(y_r)=g_1(y_{r+1})$. Then a mapping g such that $g(u)=g_0(u)$ for $u\in V({}_{q,r}\boldsymbol{G})$ and $g(u)=g_1(u)\cup\{x\}$ for $u\in V(\boldsymbol{G}_{q+1,r+1})$ is an optimal embedding of $\boldsymbol{G}$ into $\boldsymbol{Q}_X$. Difficulties occur if $\mu(\boldsymbol{G},q,r)+1<\tau(\boldsymbol{G},q+1,r+1)$ whenever $|V({}_{q,r}\boldsymbol{G})|$, $|V(\boldsymbol{G}_{q+1,r+1})|\leqslant 2^{|X|-1}$. In this case we find $i=0,1,\ldots,p-1$ such that $|V({}_{j_i-1,k_i-1}\boldsymbol{G})|$, $|V(\boldsymbol{G}_{j_i+1,k_i+1})|<2^{|X|-1}$. We divide the rung $\boldsymbol{R}_i$ between ${}_{j_i-1,k_i-1}\boldsymbol{G}$ and $\boldsymbol{G}_{j_i+1,k_i+1}$ such that we obtain nice generalized ladders with size of underlying sets less or equal to $2^{|X|-1}$. By induction hypothesis, there exist optimal embeddings g_0 and g_1 of the two nice generalized ladders into $\boldsymbol{Q}_{X\setminus\{x\}}$ for some $x\in X$ and, analogously as above, we construct an injective mapping g. To obtain an optimal embedding of $\boldsymbol{G}$ to $\boldsymbol{Q}_X$ the embeddings g_0 and g_1 must satisfy some conditions and the construction has to preserve them. To formalize these conditions we give two auxiliary definitions.

For a nice generalized ladder $\boldsymbol{G}$ and for a set X with $|V(\boldsymbol{G})|\leqslant 2^{|X|}$, we say that an eight-tuple (t_0,t_1,A,B,C,D,E,F) is $(\boldsymbol{G},X)$-*bounding* if

- (a0) t_0 is an element of the subpath $(x_{\min\{j_{p-1}+1,n\}},\ldots,x_n)$ of $\boldsymbol{P}_0$ such that if the distance between t_0 and x_n is even, then $t_0=x_n$;
- (a1) t_1 is an element of the subpath $(y_0,\ldots,y_{\max\{k_0-1,0\}})$ of $\boldsymbol{P}_1$ such that if the distance between t_1 and y_0 is even, then $t_1=y_0$;
- (a2) A,B,C,D,E, and F are subsets of X such that A,B,C, and D are pairwise distinct;
- (a3) $\Delta(A,B)$ and $\Delta(C,D)$ are singletons;
- (a4) if n is even then $\Delta(A,C)$ and $\Delta(B,D)$ are two-element sets, if n is odd then $\Delta(A,C)$ and $\Delta(B,D)$ are singletons;
- (a5) $\{E,F\}\cap\{A,D\}=\emptyset$ and $(\Delta(A,C)\setminus\Delta(A,B))\cap\Delta(C,F)=\emptyset$;
- (a6) if $t_0=x_n$ then $C=F$ else $\Delta(C,F)$ is a singleton;
- (a7) if $t_1=y_0$ then $B=E$ else $\Delta(B,E)$ is a singleton.

Then an embedding $f:\boldsymbol{G}\to\boldsymbol{Q}_X$ is called (t_0,t_1,A,B,C,D,E,F)-*embedding* if $f(x_0)=A$, $f(y_0)=B$, $f(x_n)=C$, $f(y_m)=D$, $f(t_1)=E$, and $f(t_0)=F$.

Before we prove the main result of this section, observe that if (t_0,t_1,A,B,C,D,E,F) is $(\boldsymbol{G},X)$-bounding and n is odd, then $\Delta(A,B)=\Delta(C,D)\neq\Delta(A,C)=\Delta(B,D)$ and $\Delta(A,D)=\Delta(B,C)$.

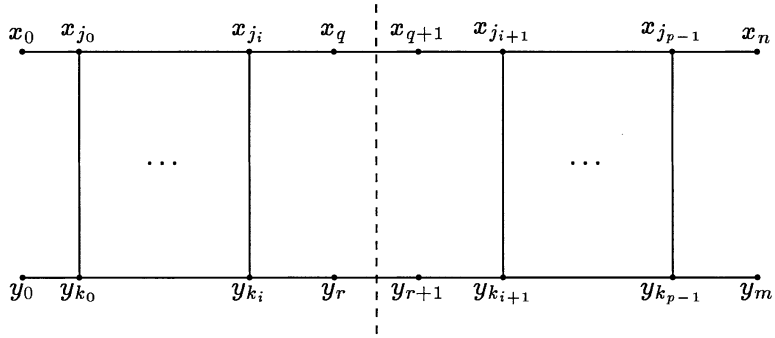

Fig. 1. The homomorphisms g_0 and g_1 satisfy $g_0(x_0) = A$, $g_0(y_0) = B$, $g_0(x_q) = G$, $g_0(y_r) = H$ and $g_1(x_{q+1}) = G$, $g_1(y_{r+1}) = H$, $g_1(x_n) = C_1$, $g_1(y_m) = D_1$.

Theorem 2.1. *Let $\boldsymbol{G}$ be a nice generalized ladder and X be a set with $|V(\boldsymbol{G})| \leqslant 2^{|X|}$. If $(t_0, t_1, A, B, C, D, E, F)$ is $(\boldsymbol{G}, X)$-bounding, then there exists a $(t_0, t_1, A, B, C, D, E, F)$-embedding f from $\boldsymbol{G}$ into $\boldsymbol{Q}_X$.*

Proof. We prove the statement by induction over $|V(\boldsymbol{G})|$. For $|V(\boldsymbol{G})| \leqslant 8$, a direct calculation shows that there exists a $(t_0, t_1, A, B, C, D, E, F)$-embedding f from $\boldsymbol{G}$ into $\boldsymbol{Q}$ for any $(\boldsymbol{G}, X)$-bounding $(t_0, t_1, A, B, C, D, E, F)$.

Now assume that $\boldsymbol{G}$ is a nice generalized ladder with $|V(\boldsymbol{G})| > 8$ and X is a set with $|V(\boldsymbol{G})| \leqslant 2^{|X|}$ and $(t_0, t_1, A, B, C, D, E, F)$ is $(\boldsymbol{G}, X)$-bounding.

Choose $x \in \Delta(A, C) \setminus (\Delta(A, B) \cup \Delta(C, D))$. Without loss of generality, we can assume that $x \in C \cap D$. Let $Y = X \setminus \{x\}$, $C_1 = C \setminus \{x\}$, $D_1 = D \setminus \{x\}$, and $F_1 = F \setminus \{x\}$. Observe that, if n is odd then $A = C_1$, $B = D_1$, if n is even then $\Delta(A, C_1)$ and $\Delta(B, D_1)$ are singletons and either $\Delta(A, B) = \Delta(C_1, D_1)$, or $A = D_1$, or $B = C_1$. For the sake of simplicity, define ${}_{-1,-1}\boldsymbol{G}$ and $\boldsymbol{G}_{n+1,m+1}$ as empty graphs and ${}_{n,m}\boldsymbol{G} = \boldsymbol{G} = \boldsymbol{G}_{0,0}$. Analogously, set $j_{-1} = 0 = k_{-1}$ and $j_p = n$, $k_p = m$. Let i be the greatest integer less than $p + 1$ such that $|V({}_{j_{i-1},k_{i-1}}\boldsymbol{G})| \leqslant 2^{|Y|}$. Then $i \geqslant 0$ and $|V(\boldsymbol{G}_{j_i+1,k_i+1})| < 2^{|Y|}$.

First assume that $|V(\boldsymbol{G}_{j_i,k_i})| \leqslant 2^{|Y|}$. Then there exist $q, r \in \mathbb{N}$ such that $j_{i-1} \leqslant q < j_i$, $k_{i-1} \leqslant r < k_i$, $q \equiv r$, and $|V({}_{q,r}\boldsymbol{G})|$, $|V(\boldsymbol{G}_{q+1,r+1})| \leqslant 2^{|Y|}$ (see Fig. 1). Set $\boldsymbol{H}_0 = {}_{q,r}\,\boldsymbol{G}$, $\boldsymbol{H}_1 = \boldsymbol{G}_{q+1,r+1}$, $t_2 = x_r$, and $t_3 = y_{r+1}$. Choose $G \subseteq Y$ with $\Delta(A, G) \cap \Delta(A, B) = \emptyset$ and such that

(A0) if n and q are odd, then $\Delta(A, G)$ is a singleton;
(A1) if n is odd and q is even, then $\Delta(A, G)$ is a two-element set;
(A2) if n is even and q is odd, then $\Delta(A, G)$ is a singleton and $G \neq C_1$;
(A3) if n and q are even, then $\Delta(C_1, G)$ is a singleton and $G \neq A$.

Set $H = \Delta(G, A, B) = \Delta(G, \Delta(A, B))$. Clearly, $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ are disjoint nice generalized ladders with $|V(\boldsymbol{H}_0)|$, $|V(\boldsymbol{H}_1)| \leqslant 2^{|Y|}$, and, by Statement 1.1(6), $V(\boldsymbol{G}) = V(\boldsymbol{H}_0) \cup V(\boldsymbol{H}_1)$. Then $(t_2, t_1, A, B, G, H, E, G)$ is $(\boldsymbol{H}_0, Y)$-bounding and $(t_0, t_3, G, H, C_1, D_1, H, F_1)$ is $(\boldsymbol{H}_1, Y)$-bounding because if n is even then q is odd just when $n - q - 1$ is even.

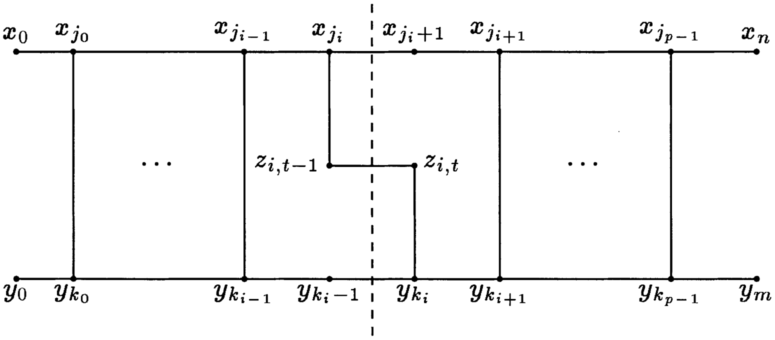

Fig. 2. The homomorphisms g_0 and g_1 satisfy $g_0(x_0) = A$, $g_0(y_0) = B$, $g_0(x_{j_i}) = H_2$, $g_0(z_{i,t-1}) = G$, $g_0(y_{k_{i-1}}) = H_1$ and $g_1(x_{j_i+1}) = H_2$, $g_1(z_{i,t}) = G$, $g_1(y_{k_i}) = H_1$, $g_1(x_n) = C_1$, $g_1(y_m) = D_1$.

Thus, by induction hypothesis, there exist a $(t_2, t_1, A, B, G, H, E, G)$-embedding g_0 of $\boldsymbol{H}_0$ into $\boldsymbol{Q}_Y$ and a $(t_0, t_3, G, H, C_1, D_1, H, F_1)$-embedding g_1 of $\boldsymbol{H}_1$ into $\boldsymbol{Q}_Y$ (see Fig. 1).

Secondly, assume that $|V(\boldsymbol{G}_{j_i,k_i})| > 2^{|Y|}$. Then $|V({}_{j_i-1,k_i-1}\boldsymbol{G})| < 2^{|Y|}$. Then there exists an even $t \in \mathbb{N}$ with $t < \ell_i$ and $|V({}_{j_i-1,k_i-1}\boldsymbol{G})| + t$, $|V(\boldsymbol{G}_{j_i+1,k_i+1})| + \ell_i - t \leqslant 2^{|Y|}$. Let $\boldsymbol{H}_0$ be an induced subgraph of $\boldsymbol{G}$ on the set $V({}_{j_i-1,k_i-1}\boldsymbol{G}) \cup \{z_{i,h} \mid h = 0, 1, \ldots, t-1\}$ and $\boldsymbol{H}_1$ be an induced subgraph of $\boldsymbol{G}$ on the set $V(\boldsymbol{G}_{j_i+1,k_i+1}) \cup \{z_{i,h} \mid h = t,\ t+1, \ldots, \ell_i\}$ (see Fig. 2). Set $t_2 = x_{j_i}$, and $t_3 = y_{k_i}$; Then the distances between t_2 and $z_{i,t-1}$ in $\boldsymbol{H}_0$ and between t_3 and $z_{i,t}$ in $\boldsymbol{H}_1$ are odd. Therefore the distance between x_0 and z_{i,t_1} in the leading path of $\boldsymbol{H}_0$ and $k_i - 1$ have the same parity and the distance between $z_{i,t}$ and y_m in the leading path of $\boldsymbol{H}_1$ and $n - j_i - 1$ have the same parity. Hence $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ are disjoint nice generalized ladders with $|V(\boldsymbol{H}_0)|$, $|V(\boldsymbol{H}_1)| \leqslant 2^{|Y|}$, and $V(\boldsymbol{G}) = V(\boldsymbol{H}_0) \cup V(\boldsymbol{H}_1)$. Choose $G \subseteq Y$ such that $\Delta(A, G) \cap \Delta(A, B) = \emptyset = \Delta(A, G) \cap \Delta(B, E)$ and

(A4) if n and $k_i - 1$ are odd then $\Delta(A, G)$ is a singleton;
(A5) if n is odd and $k_i - 1$ is even then $\Delta(A, G)$ is a two element set;
(A6) if n is even and $k_i - 1$ is odd then $\Delta(A, G)$ is a singleton and $G \neq C_1$;
(A7) if n and $k_i - 1$ are even then $\Delta(C_1, G)$ is a singleton and $G \neq A$.

If n is odd then set $H_1 = \Delta(G, A, B) = \Delta(G, \Delta(A, B))$ and if $k_i - 1$ is odd choose $H_2 \subseteq Y$ with $H_2 \neq H_1$, A and such that $\Delta(H_2, G)$ is a singleton, if $k_i - 1$ is even then choose $H_2 \subseteq Y$ such that $\Delta(A, H_2)$ and $\Delta(G, H_2)$ are singletons (thus $H_2 \neq H_1$).

Assume that n is even. If $k_i - 1$ is odd then set $H_1 = \Delta(G, A, B) = \Delta(G, \Delta(A, B))$ and

$$H_2 = \begin{cases} \Delta(G, A, C_1) = \Delta(G, \Delta(A, C_1)) & \text{if } |\{A, B, C_1, D_1\}| = 4, \\ \Delta(G, C_1, D_1) = \Delta(G, \Delta(C_1, D_1)) & \text{if } |\{A, B, C_1, D_1\}| = 3. \end{cases}$$

If $k_i - 1$ is even then set

$$H_1 = \begin{cases} \Delta(G, A, C_1) = \Delta(G, \Delta(A, C_1)) & \text{if } |\{A, B, C_1, D_1\}| = 4, \\ \Delta(G, A, B) = \Delta(G, \Delta(A, B)) & \text{if } |\{A, B, C_1, D_1\}| = 3, \end{cases}$$

$$H_2 = \begin{cases} \Delta(G, A, B) = \Delta(G, \Delta(A, B)) & \text{if } |\{A, B, C_1, D_1\}| = 4, \\ \Delta(G, C_1, D_1) = \Delta(G, \Delta(C_1, D_1)) & \text{if } |\{A, B, C_1, D_1\}| = 3. \end{cases}$$

Then $(t_2, t_1, A, B, G, H_1, E, H_2)$ is $(\boldsymbol{H}_0, Y)$-bounding and $(t_0, t_3, H_2, G, C_1, D_1, H_1, F_1)$ is $(\boldsymbol{H}_1, Y)$-bounding. Thus, by induction hypothesis, there exist a $(t_2, t_1, A, B, G, H_1, E, H_2)$-embedding g_0 of $\boldsymbol{H}_0$ into $\boldsymbol{Q}_Y$ and a $(t_0, t_3, H_2, G, C_1, D_1, H_1, F_1)$-embedding g_1 of $\boldsymbol{H}_1$ into $\boldsymbol{Q}_Y$ (see Fig. 2).

In both cases define a mapping g such that

$$g(u) = \begin{cases} g_0(u) & \text{if } u \in V(\boldsymbol{H}_0), \\ g_1(u) \cup \{x\} & \text{if } u \in V(\boldsymbol{H}_1). \end{cases}$$

Direct verification shows that g is a $(t_0, t_1, A, B, C, D, E, F)$-embedding of $\boldsymbol{G}$ into $\boldsymbol{Q}_X$. $\square$

Since any ladder with even rungs is a nice generalized ladder the following consequence of Theorem 2.1 generalizes the result of [2]. Corollary 2.2 also gives the affirmative solution of Havel conjecture for balanced bipartite generalized ladders with even rungs and for balanced bipartite cyclic generalized ladders with even rungs.

Corollary 2.2. *The following claims are true*:

(1) *Any nice generalized ladder has an optimal embedding into a hypercube.*
(2) *Any bipartite cyclic generalized ladder with even rungs has an optimal embedding into a hypercube.*
(3) *Any balanced bipartite generalized ladder with even rungs has an optimal embedding into a hypercube.*

Proof. (1) immediately follows from Theorem 2.1. Indeed, for a nice generalized ladder $\boldsymbol{G}$ take a hypercube $\boldsymbol{Q}_X$ over a set X with $2^{|X|-1} < |V(\boldsymbol{G})| \leqslant 2^{|X|}$; it is clear that such a set X exists. Set $t_0 = n$, $t_1 = 0$ and choose $A, B, C, D, E, F, \subseteq X$ so that $B = E$, $C = F, A, B, C$, and D are pairwise distinct, $\Delta(A,B) = \Delta(C,D)$ is a singleton, $0 < |\Delta(A,C)|, |\Delta(B,D)| < 3$, and $n \equiv |\Delta(A,C)| \equiv |\Delta(B,D)|$. Then $(t_0, t_1, A, B, C, D, E, F)$ is $(\boldsymbol{G}, X)$-bounding. Since such sets exist, Theorem 2.1 completes the proof of (1).

To prove (2), we combine Statement 1.2(3) and Theorem 2.1. For a bipartite cyclic generalized ladder $\boldsymbol{H}$ there exists, by Statement 1.2(3), a nice generalized ladder $\boldsymbol{G}$ such that the lengths of leading paths of $\boldsymbol{G}$ are odd and $\boldsymbol{H} = \text{cyclic}\ (\boldsymbol{G})$. Take a set X with $2^{|X|-1} < |V(\boldsymbol{G})| = |V(\boldsymbol{H})| \leqslant 2^{|X|}$ and apply Theorem 2.1 to a $(\boldsymbol{G}, X)$-bounding family to complete the proof of (2).

If $\boldsymbol{G}$ is a balanced bipartite generalized ladder then, by Statements 1.1(2) and 1.2(1), $\boldsymbol{H} = \text{cyclic}(\boldsymbol{G})$ is a bipartite cyclic generalized ladder, and (3) follows from (2). $\square$

3. Generalized ladders with odd rungs

The aim of this section is to find a class of generalized ladders with odd rungs that have optimal embeddings. A computer program found a ladder with short odd rungs (any of its rungs has three or five vertices) which does not have an optimal embedding, see [4]. The main obstacle to a construction based on the idea from the second section is that it is unknown whether the class of generalized ladders with odd rungs is closed under 'halving of its members', as this idea requires. We thus apply another idea, one which was first used in [1]. Results from [4] were obtained by a modification of this idea from [1], and our construction generalizes the method from [4]. We now describe the idea of our construction.

Assume that $\boldsymbol{G}$ is a generalized ladder with odd rungs. We shall construct, by induction over number of rungs, a $\mathscr{D}$-embedding of $\boldsymbol{G}$ into a hypercube $\boldsymbol{Q}$ over a set X with $|V(\boldsymbol{G})| \leqslant 2^{|X|}$ where $\mathscr{D}$ is a dense set of $\boldsymbol{Q}$. Let $\boldsymbol{G}_1 = _{j_{p-2}-1,k_{p-2}-1}\boldsymbol{G}$ and $\boldsymbol{G}_2 = \boldsymbol{G}_{j_{p-2},k_{p-2}}$. Then, by Statement 1.3(1) and (5), there exist disjoint dense sets $\mathscr{D}_1, \mathscr{D}_2$ and $\mathscr{D}_3$ with $|\mathscr{D}_1| = |V(\boldsymbol{G}_1)|, |\mathscr{D}_2 \cup \mathscr{D}_3| = |V(\boldsymbol{G}_2)|, \mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_3 = \mathscr{D}$ and such that $\mathscr{D}_1 \cup \mathscr{D}_2$ is a dense set and the greatest power of 2 such that $|\mathscr{D}_1 \cup \mathscr{D}_2|$ is its multiple is greater than $|\mathscr{D}_3|$. By the induction hypothesis, there exists a $\mathscr{D}_1$-embedding g of $\boldsymbol{G}_1$ and we construct a $\mathscr{D}_2 \cup \mathscr{D}_3$-embedding h of $\boldsymbol{G}_2$ such that $\{g(x_{j_{p-2}-1}),\ h(x_{j_{p-2}})\}$ and $\{g(y_{k_{p-2}-1}),\ h(y_{k_{p-2}})\}$ are edges of $\boldsymbol{Q}$. Hence there exists a $\mathscr{D}$-embedding of $\boldsymbol{G}$. Now we give the outline of the proof. The precise formal proof is too long and tiresome.

First, we define a class of generalized ladders with odd rungs for which our construction does work. A generalized ladder $\boldsymbol{G}$ with odd rungs is called pretty if it satisfies:

(b0) n and m are odd and j_0 and p are even;
(b1) $j_i \equiv k_i$ for all $i = 0, 1, \ldots, p-1$ and $j_i \not\equiv j_{i+1}$ for all $i = 0, 1, \ldots, p-2$;
(b2) either $j_0 k_0 = 0$ or $j_0 + k_0 \geqslant 10$;
(b3) for every $i = 0, 1, \ldots, p/2 - 1$ one of the following three conditions is true:
  (A) if $j_{2i+2} = j_{2i+1} + 1$ and $k_{2i+2} = k_{2i+1} + 1$, then for $r, q \in \mathbb{N}$ such that r is odd, $|V(_{j_{2i}-1,k_{2i}-1}\boldsymbol{G})| \leqslant r2^q$ and $0 \leqslant |V(_{j_{2i+2}-1,k_{2i+2}-1}\boldsymbol{G})| - r2^q < 2^q$, we have $|V(_{j_{2i+2}-1,k_{2i+2}-1}\boldsymbol{G})| - r2^q \geqslant 8$ or $r2^q - |V(_{j_{2i}-1,k_{2i}-1}\boldsymbol{G})| \geqslant 8$;
  (B) if $j_{2i+1} = j_{2i} + 1, k_{2i+1} = k_{2i} + 1, \ell_{2i} = \ell_{2i+1} = 2$ and either $j_{2i+2} = j_{2i+1} + 1$ or $k_{2i+2} = k_{2i+1} + 1$, then for $r, q \in \mathbb{N}$ such that r is odd, $|V(_{j_{2i}-1,k_{2i}-1}\boldsymbol{G})| \leqslant r2^q$ and $0 \leqslant |V(_{j_{2i+2}-1,k_{2i+2}-1}\boldsymbol{G})| - r2^q < 2^q$, we have $|V(_{j_{2i+2}-1,k_{2i+2}-1}\boldsymbol{G})| - r2^q \geqslant 8$ or $r2^q - |V(_{j_{2i}-1,k_{2i}-1}\boldsymbol{G})| \geqslant 8$;
  (C) for positive integers r and q such that r is odd, $|V(_{j_{2i}-1,k_{2i}-1}\boldsymbol{G})| \leqslant r2^q$ and $0 \leqslant |V(_{j_{2i+2}-1,k_{2i+2}-1}\boldsymbol{G})| - r2^q < 2^q$, we have $|V(_{j_{2i+2}-1,k_{2i+2}-1}\boldsymbol{G})| - r2^q \geqslant 16$ or $r2^q - |V(_{j_{2i}-1,k_{2i}-1}\boldsymbol{G})| \geqslant 16$.

Observe that for even $s, t \in \mathbb{N}$, there exist unique $r, q \in \mathbb{N}$ such that r is odd, $s \leqslant r2^q$ and $0 \leqslant s + t - r2^q < 2^q$ and thus the r and q from conditions (b3A), (b3B) and (b3C) exist and are uniquely determined.

The conditions (b0) and (b1) ensure that $\boldsymbol{G}$ is a balanced bipartite graph, see Statement 1.1(3), the condition (b2) ensures the initial step of our construction, and the condition (b3) ensures the $\mathscr{D}_2 \cup \mathscr{D}_3$-embedding of $_{j_{2i+2}-1,k_{2i+2}-1}\boldsymbol{G}_{j_{2i},k_{2i}}$, see the begining of this section. We note that the condition (b3A) implies that any cellular ladder (for the definition, see [4]) is a pretty generalized ladder, and hence Theorem 3.4

generalizes Theorem 5.1 from [4]. Our construction exploits the specific configurations of paths and cycles in a hypercubes, and we define these next.

For a dense set $\mathscr{A}$ and an edge $e \in E(\boldsymbol{Q})$ with $e \subseteq \mathscr{A}$ we say that $(\mathscr{A}, e)$ is *good* if $|\mathscr{A}| \neq 2^r + 2$ for each $r \in \mathbb{N}$ with $r > 1$ or there exists a canonical decomposition $(\mathscr{A}_0, \mathscr{A}_1)$ of $\mathscr{A}$ such that $\Delta(e,x) \neq \mathscr{A}_0$ for all $x \in X \setminus \mathrm{dir}(e)$.

Let $\boldsymbol{C}$ be a cycle. Then we say that $\boldsymbol{C}$ is an (n,e)-cycle if n is length of $\boldsymbol{C}$ and either $n > 0$ and $e \in E(\boldsymbol{C})$ or $n = 0$ and e is an arbitrary edge. A graph $\boldsymbol{G}$ is called a *doublecycle* if it is a union of two disjoint cycles, there exist precisely two cycles $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$ with $V(\boldsymbol{C}_0) \cap V(\boldsymbol{C}_1) = \emptyset, V(\boldsymbol{G}) = V(\boldsymbol{C}_0) \cup V(\boldsymbol{C}_1)$ and $E(\boldsymbol{G}) = E(\boldsymbol{C}_0) \cup E(\boldsymbol{C}_1)$. For $e_0, e_1 \in E(\boldsymbol{G})$ we shall write $e_0 \cong_{\boldsymbol{G}} e_1$ if $e_0, e_1 \in E(\boldsymbol{C}_i)$ for some $i = 0, 1$. If $\boldsymbol{C}_i$ is an (n_i, e_i)-cycle for $i = 0, 1$, then we say that $\boldsymbol{G}$ is an (n_0, n_1, e_0, e_1)-*doublecycle* ($\boldsymbol{G}$ is also an (n_1, n_0, e_1, e_0)-doublecycle). If $\boldsymbol{C}$ is an (n,e)-cycle then $\boldsymbol{C}$ is an $(n, 0, e, e')$-doublecycle and also a $(0, n, e', e)$-doublecycle for an arbitrary edge e'. We say that a cycle or a doublecycle $\boldsymbol{C}$ is $\mathscr{A}$-*spanning* for a set $\mathscr{A}$ if $\mathscr{A} = V(\boldsymbol{C})$. We recall results from [4].

Statement 3.1 (Caha and Koubek [4]). *Let $\mathscr{A}$ be a dense set of a hypercube $\boldsymbol{Q}$ over a set X with a canonical decomposition $(\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_n)$ and an even size. Then*

(1) *if $e \in E(\boldsymbol{Q})$ is an edge then there exists an $\mathscr{A}$-spanning cycle $\boldsymbol{C}$ with $e \in E(\boldsymbol{C})$ if and only if $(\mathscr{A}, e)$ is good*;
(2) *if $|\mathscr{A}_0| \geqslant 4$ then for every family $\{e_i, e_i'\}_{i=0}^n$* of edges of $E(\boldsymbol{Q})$ *with $e_i, e_i' \subseteq \mathscr{A}_i$ for all $i = 0, 1, \ldots, n$ there exists an $\mathscr{A}$-spanning cycle $\boldsymbol{C}$ with $e_i, e_i' \in E(\boldsymbol{C})$ for all $i = 0, 1, \ldots, n$*;
(3) *if $n = 0$ and $|\mathscr{A}_0| \geqslant 4$, then for every triple of pairwise distinct edges e_0, e_1 and e_2 with $e_0, e_1, e_2 \subseteq \mathscr{A}$ and $e_0 \cap e_1 \cap e_2 = \emptyset$ there exists an $\mathscr{A}$-spanning cycle $\boldsymbol{C}$ in $\boldsymbol{Q}$ with $e_0, e_1, e_2 \in E(\boldsymbol{C})$*;
(4) *if $e_0, e_1 \in E(\boldsymbol{Q})$ are disjoint edges with $e_0, e_1 \subseteq \mathscr{A}_0$, then for even $n_0, n_1 \in \mathbb{N}$ with $n_0 + n_1 = |\mathscr{A}|$ there exists an $\mathscr{A}$-spanning (n_0, n_1, e_0, e_1)-doublecycle $\boldsymbol{G}$ of $\boldsymbol{Q}$.*

A generalization of Statement 3.1 used in our construction is based on an easy technique of merging cycles and doublecycles, see [4].

Let $\boldsymbol{Q}$ be a hypercube over a set X. We say that disjoint edges $e_0, e_1 \in E(\boldsymbol{Q})$ are *parallel* whenever $\Delta(e_0, x) = e_1$ for some $x \in X$ (thus subcubes e_0 and e_1 are x-brothers). For parallel edges e_0 and e_1 in $\boldsymbol{Q}$, let $\complement(e_0, e_1)$ denote the unique parallel pair $\{e_2, e_3\}$ of edges in $\boldsymbol{Q}$ with $\{e_0, e_1\} \neq \{e_2, e_3\}$ and $e_0 \cup e_1 = e_2 \cup e_3$ (this is equivalent to the fact that there exists a cycle $\boldsymbol{C}$ of length 4 with $E(\boldsymbol{C}) = \{e_0, e_1, e_2, e_3\}$).

Let $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be disjoint (n_1, e_1)- and/or (n_2, e_2)-cycles of $\boldsymbol{Q}$ such that n_1 and n_2 are even and e_1, e_2 are parallel. For $\{e_3, e_4\} = \complement(e_1, e_2)$ define a graph $\boldsymbol{C} = (\boldsymbol{C}_1 \oplus \boldsymbol{C}_2 / \{e_3, e_4\})$ so that

- if $n_1 = 0$, then $\boldsymbol{C} = \boldsymbol{C}_2$;
- if $n_2 = 0$, then $\boldsymbol{C} = \boldsymbol{C}_1$;
- if $n_1 = n_2 = 2$, then $V(\boldsymbol{C}) = V(\boldsymbol{C}_1) \cup V(\boldsymbol{C}_2)$ and $E(\boldsymbol{C}) = \{e_1, e_2, e_3, e_4\}$;

- if $n_1 = 2$ and $n_2 \geqslant 4$, then $V(\boldsymbol{C}) = V(\boldsymbol{C}_1) \cup V(\boldsymbol{C}_2)$ and

  $E(\boldsymbol{C}) = (E(\boldsymbol{C}_2) \setminus \{e_2\}) \cup \{e_1, e_3, e_4\}$;
- if $n_1 \geqslant 4$ and $n_2 = 2$, then $V(\boldsymbol{C}) = V(\boldsymbol{C}_1) \cup V(\boldsymbol{C}_2)$ and

  $E(\boldsymbol{C}) = (E(\boldsymbol{C}_1) \setminus \{e_1\}) \cup \{e_2, e_3, e_4\}$;
- if $n_1, n_2 \geqslant 4$, then $V(\boldsymbol{C}) = V(\boldsymbol{C}_1) \cup V(\boldsymbol{C}_2)$ and

  $E(\boldsymbol{C}) = (E(\boldsymbol{C}_1) \cup E(\boldsymbol{C}_2) \cup \{e_3, e_4\}) \setminus \{e_1, e_2\}$.

In any case, $\boldsymbol{C}$ is a cycle of length $n_1 + n_2$.

Analogously, let $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ be disjoint $(n_1,\ n_2,\ e_1,\ e_2)$- and (n_3, n_4, e_3, e_4)-doublecycles of $\boldsymbol{Q}$ such that n_1, n_2, n_3, and n_4 are even and $\{e_1, e_3\}$ and $\{e_2, e_4\}$ are parallel pairs. Let $\{e_5, e_6\} = ʗ(e_1, e_3)$ and $\{e_7, e_8\} = ʗ(e_2, e_4)$. If $\boldsymbol{C}_1,\ \boldsymbol{C}_2,\ \boldsymbol{C}_3$ and $\boldsymbol{C}_4$ are pairwise disjoint (n_1, e_1)-, (n_2, e_2)-, (n_3, e_3)- and/or (n_4, e_4)-cycles with $\boldsymbol{G}_1 = \boldsymbol{C}_1 \cup \boldsymbol{C}_2$ and $\boldsymbol{G}_2 = \boldsymbol{C}_3 \cup \boldsymbol{C}_4$, then define

$$\boldsymbol{G} = (\boldsymbol{G}_1 \oplus \boldsymbol{G}_2 / \{e_5, e_6\}, \{e_7, e_8\}) = (\boldsymbol{C}_1 \oplus \boldsymbol{C}_3 / \{e_5, e_6\}) \cup (\boldsymbol{C}_2 \oplus \boldsymbol{C}_4 / \{e_7, e_8\}).$$

Then $\boldsymbol{G}$ is a disjoint union of cycles of lengths $n_1 + n_3$ and $n_2 + n_4$.

Let $\boldsymbol{G}$ be an (n_1, n_2, e_1, e_2)-doublecycle of $\boldsymbol{Q}$ and let $\boldsymbol{C}$ be a (n_3, e_3)-cycle of $\boldsymbol{Q}$ disjoint with $\boldsymbol{G}$ such that n_1, n_2 and n_3 are even and e_1, e_3 are parallel. Let $\{e_4, e_5\} = ʗ(e_1, e_3)$. If $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are disjoint (n_1, e_1)- and/or (n_2, e_2)-cycles with $\boldsymbol{G} = \boldsymbol{C}_1 \cup \boldsymbol{C}_2$, then define

$$\boldsymbol{G}' = (\boldsymbol{G} \oplus \boldsymbol{C} / \{e_5, e_6\}) = (\boldsymbol{C}_1 \oplus \boldsymbol{C} / \{e_5, e_6\}) \cup \boldsymbol{C}_2.$$

Clearly, $\boldsymbol{G}'$ is a disjoint union of cycles of lengths $n_1 + n_3$ and n_2.

We say that two disjoint dense sets $\mathscr{A}$ and $\mathscr{B}$ of a hypercube $\boldsymbol{Q}$ are *a-connected* for $a \in X$ if there exist canonical decompositions $(\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_n)$ of $\mathscr{A}$ and $(\mathscr{B}_0, \mathscr{B}_1, \ldots, \mathscr{B}_m)$ of $\mathscr{B}$ and a subcube $\mathscr{C}$ with $|\mathscr{C}| = 8$, $|\mathscr{C} \cap \mathscr{A}_n| = |\mathscr{C} \cap \mathscr{B}_m| = 4$, and $a \in \operatorname{dir}(\mathscr{A}_n) \cap \operatorname{dir}(\mathscr{B}_m) \cap \operatorname{dir}(\mathscr{C})$.

Statement 3.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be disjoint a-connected dense sets of a hypercube $\boldsymbol{Q}$ over a set X for some $a \in X$ with canonical decompositions $(\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_m)$ and $(\mathscr{B}_0, \mathscr{B}_1, \ldots, \mathscr{B}_q)$. Let n_0, n_1, n_2, and n_3 be even nonnegative integers. Assume that $e_0, e_1 \in E(\boldsymbol{Q})$ are disjoint edges with $e_0, e_1 \subseteq \mathscr{A}_l$ for some $l \in \{0, 1, \ldots, m\}$ such that $(\mathscr{A}, e_0)$ and $(\mathscr{A}, e_1)$ are good.*

(1) *If $a \notin \mathscr{A}_0 \cup \operatorname{dir}(e_0)$ then for every edge $e \in E(\boldsymbol{Q})$ such that $(\mathscr{B}, e)$ is good there exists an $\mathscr{A} \cup \mathscr{B}$-spanning cycle $\boldsymbol{C}$ with e_0, $e \in E(\boldsymbol{C})$.*

(2) *If $l = 0$, $n_0 + n_1 = |\mathscr{A}|$ and there exist edges $e_2, e_3 \in E(\boldsymbol{Q})$ and $b \in X$ such that $e_{i+2} \subseteq \mathscr{A}_{s(i)}$ for the greatest number $s(i)$ with $n_i > \sum_{j=0}^{s(i)} |\mathscr{A}_j|/2$ for $i = 0, 1$ and*

   $$b \in (\min(e_0 \cup e_2) \setminus \max(e_1 \cup e_3)) \cup (\min(e_1 \cup e_3) \setminus \max(e_0 \cup e_2)),$$

   then there exists an $\mathscr{A}$-spanning (n_0, n_1, e_0, e_1)-doublecycle $\boldsymbol{G}$ with $e_i \cong_{\boldsymbol{G}} e_{i+2}$ for $i = 0, 1$.

(3) *If* $n_0+n_1=|\mathscr{A}|$ *then there exists an* $\mathscr{A}$*-spanning* (n_0,n_1,e_0,e_1)*-doublecycle.*
(4) *If* $n_0+n_1=|\mathscr{A}|\cup|\mathscr{B}|$ *and* $a\in(\min(e_0)\backslash\max(e_1)\cup(\min(e_1))\backslash\max(e_0))$, *then there exists an* $\mathscr{A}\cup\mathscr{B}$*-spanning* (n_0,n_1,e_0,e_1)*-doublecycle.*
(5) *If there are disjoint edges* $e_2,e_3\in E(\boldsymbol{Q})$ *with* $e_2,e_3\subseteq\mathscr{A}_{l'}$ *for some* $l'\in\{0,1,\dots,m\}$,

$$(\min(e_0\cup e_1)\backslash\max(e_2\cup e_3))\cup(\min(e_2\cup e_3)\backslash\max(e_0\cup e_1))\neq\emptyset,$$

and such that $(\mathscr{A},e_2)$ *and* $(\mathscr{A},e_3)$ *are good, then there exist an* (n_0,n_1,e_0,e_1)*-doublecycle* $\boldsymbol{G}_1$ *and an* (n_2,n_3,e_2,e_3)*-doublecycle* $\boldsymbol{G}_2$ *with* $V(\boldsymbol{G}_1)\cap V(\boldsymbol{G}_2)=\emptyset$ *and* $V(\boldsymbol{G}_1)\cup V(\boldsymbol{G}_2)=\mathscr{A}$ *whenever* $n_0+n_1+n_2+n_3=|\mathscr{A}|$.
(6) *If there are disjoint edges* $e_2,e_3\in E(\boldsymbol{Q})$ *with* $e_2,e_3\subseteq\mathscr{A}_{l'}$ *for some* $l'\in\{0,1,\dots,m\}$,

$$(\min(e_0\cup e_1)\backslash\max(e_2\cup e_3))\cup(\min(e_2\cup e_3)\backslash\max(e_0\cup e_1))\neq\emptyset$$

and

$$a\in((\min(e_0)\backslash\max(e_1))\cup(\min(e_1)\backslash\max(e_0)))\cap((\min(e_2)\backslash\max(e_3))$$
$$\cup(\min(e_3)\backslash\max(e_2))),$$

and such that $(\mathscr{A},e_2)$ *and* $(\mathscr{A},e_3)$ *are good, then there exist an* (n_0,n_1,e_0,e_1)*-doublecycle* $\boldsymbol{G}_1$ *and an* (n_2,n_3,e_2,e_3)*-doublecycle* $\boldsymbol{G}_2$ *with* $V(\boldsymbol{G}_1)\cap V(\boldsymbol{G}_2)=\emptyset$ *and* $V(\boldsymbol{G}_1)\cup V(\boldsymbol{G}_2)=\mathscr{A}\cup\mathscr{B}$ *whenever* $n_0+n_1+n_2+n_3=|\mathscr{A}|+|\mathscr{B}|$.

Proof. Let $\mathscr{C}$ be a subcube with $|\mathscr{C}|=8$ and $|\mathscr{C}\cap\mathscr{A}_m|=|\mathscr{C}\cap\mathscr{B}_q|=4$. Then there exists $b\in X$ with $\Delta(\mathscr{C}\cap\mathscr{A}_m,b)=\mathscr{C}\cap\mathscr{B}_q$. By Statement 3.1(1) and (2), there exist an $\mathscr{A}$-spanning cycle $\boldsymbol{C}_0$ and an edge $e'\in E(\boldsymbol{Q})$ with $e'\subseteq\mathscr{C}\cap\mathscr{A}_m$ and $e_0,e'\in E(\boldsymbol{C}_0)$. By Statement 3.1(1), there exists a $\mathscr{B}$-spanning cycle $\boldsymbol{C}_1$ with $e,\Delta(e',b)\in E(\boldsymbol{C}_1)$. Then $(\boldsymbol{C}_0\oplus\boldsymbol{C}_1/\complement(e',\Delta(e',b)))$ completes the proof of (1).

We prove (2) by induction over $|\mathscr{A}|$. Since the initial step is straightforward, we prove only the induction step. With no loss of generality, we may assume that $n_0\geqslant n_1$ and $b\in(\min(e_0\cup e_2)\backslash(\max(e_1\cup e_3))$. Let $\mathscr{E}=\{A\in\mathscr{A}\mid b\in A\}$, $\mathscr{D}=\{A\in\mathscr{A}\mid b\notin A\}$, $\mathscr{E}_i=\{A\in\mathscr{A}_i\mid b\in A\}$, $\mathscr{D}_i=\{A\in\mathscr{A}_i\mid b\notin A\}$ for all $i=0,1,\dots,m$. Clearly, $\mathscr{E}$ and $\mathscr{D}$ are dense sets with canonical decompositions $(\mathscr{E}_0,\mathscr{E}_1,\dots,\mathscr{E}_m)$ and $(\mathscr{D}_0,\mathscr{D}_1,\dots,\mathscr{D}_m)$, and $|\mathscr{E}|=|\mathscr{D}|=|\mathscr{A}|/2$ and $|\mathscr{E}_i|=|\mathscr{D}_i|=|\mathscr{A}_i|/2$ for all $i=0,1,\dots,m$ because $b\in\operatorname{dir}(\mathscr{A}_0)$. Clearly, $e_0,e_2\subseteq\mathscr{E}$ and $e_1,e_3\subseteq\mathscr{D}$. By Statement 3.1(2), there exists an $\mathscr{E}$-spanning cycle $\boldsymbol{C}_0$ of $\boldsymbol{Q}$ with $e_0,e_2\in E(\boldsymbol{C}_0)$. Let k denote the greatest number with $\sum_{i=0}^{k}|\mathscr{D}_i|<n_1$ (if such k does not exist, set $k=-1$). Set $\mathscr{D}'=\bigcup_{i=0}^{k}\mathscr{D}_i$, then, by Statement 1.3(3), $\mathscr{D}'$ is a dense set. By Statement 1.3(1), there exists a dense set $\mathscr{D}''\subseteq\mathscr{D}_{k+1}$ with $|\mathscr{D}''|=n_1-|\mathscr{D}'|$, $e_3\subseteq\mathscr{D}''$, and dense sets $\mathscr{D}'$ and $\mathscr{D}''$ satisfy the assumptions of (1). By (1), there exists a $\mathscr{D}'\cup\mathscr{D}''$-spanning cycle $\boldsymbol{C}_1$ with $e_1,e_3\in E(\boldsymbol{C}_1)$. By Statement 1.3(2), $\mathscr{D}'''=\mathscr{D}_{k+1}\backslash\mathscr{D}''$ is a dense set and, by Statement 1.3(4), $\mathscr{F}=\mathscr{D}'''\cup(\bigcup_{i=k+2}^{m}\mathscr{D}_i)$ is a dense set. Clearly, we can choose $e\in E(\boldsymbol{C}_0)$ with $e\subseteq\mathscr{E}_m$ and, by Statement 3.1(1), there exists an $\mathscr{F}$-spanning cycle $\boldsymbol{C}_2$ with $\Delta(e,b)\in E(\boldsymbol{C}_2)$. Setting $\boldsymbol{C}_3=(\boldsymbol{C}_0\oplus\boldsymbol{C}_2/\complement(e,\Delta(e,b)))$ and $\boldsymbol{G}=\boldsymbol{C}_1\cup\boldsymbol{C}_3$ completes the proof of (2).

We prove (3), by induction over m. If $m=0$ then, by Statement 3.1(4), the assertion is proved. Assume that $m>0$ and $n_0\geqslant n_1$. Let $b\in X$ with $\Delta(\mathscr{A}_{m-1},b)\subseteq\mathscr{A}_m$, then

$\Delta(\mathscr{A}_i,b)\subseteq\mathscr{A}_m$ for all $i=0,1,\dots,m-1$. Set $\mathscr{A}'=\bigcup_{i=0}^{m-1}\mathscr{A}_i$ then $\mathscr{A}'$ and $\mathscr{A}_m$ are dense sets. First assume that $l<m$. Then set $n_3=\min\{n_1,|\mathscr{A}'|/2\}$ and $n_2=|\mathscr{A}'|-n_3$. By the induction hypothesis, there exists an $\mathscr{A}'$-spanning (n_2,n_3,e_0,e_1)-doublecycle $\boldsymbol{G}_0$ of $\boldsymbol{Q}$. Choose edges $e_2,e_3\in E(\boldsymbol{G}_0)$ with $e_0\cong_{\boldsymbol{G}_0}e_2$, $e_1\cong_{\boldsymbol{G}_0}e_3$ and such that $n_i>2$ implies $e_i\neq e_{i-2}$ for $i=2,3$. Then for $e_4=\Delta(e_2,b)$, $e_5=\Delta(e_3,b)$, by Statement 3.1(4), there exists an $\mathscr{A}_m$-spanning $(n_0-n_2,n_1-n_3,e_4,e_5)$-doublecycle $\boldsymbol{G}_1$ because $e_4,e_5\subseteq\mathscr{A}_m$. Then $\boldsymbol{G}=(\boldsymbol{G}_0\oplus\boldsymbol{G}_1/\complement(e_2,e_4),\complement(e_3,e_5))$ is an $\mathscr{A}$-spanning (n_0,n_1,e_0,e_1)-doublecycle. Secondly assume that $l=m$. Then set $n_3=\min\{n_1,|\mathscr{A}_n|/2\}$ and $n_2=|\mathscr{A}_n|-n_3$. By Statement 3.1(4), there exists an $\mathscr{A}_m$-spanning (n_2,n_3,e_0,e_1)-doublecycle $\boldsymbol{G}_0$. Choose edges $e_2,e_3\in E(\boldsymbol{G})$ with $e_0\cong_{\boldsymbol{G}_0}e_2$, $e_1\cong_{\boldsymbol{G}_0}e_3$, $e_4=\Delta(e_2,b)$, $e_5=\Delta(e_3,b)\subseteq\mathscr{A}_{m-1}$, and such that $n_i>2$ implies $e_i\neq e_{i-2}$ for $i=2,3$. By the induction hypothesis, there exists an $\mathscr{A}'$-spanning $(n_0-n_2,n_1-n_3,e_4,e_5)$-doublecycle $\boldsymbol{G}_1$. Then $\boldsymbol{G}=(\boldsymbol{G}_0\oplus\boldsymbol{G}_1/\complement(e_2,e_4),\complement(e_3,e_5))$ is an $\mathscr{A}$-spanning (n_0,n_1,e_0,e_1)-doublecycle and the proof of (3) is complete.

The proof of (4) proceeds in the same way, and therefore we omit it.

We prove (5), by induction over $|\mathscr{A}|$. With no loss of generality, we may assume that $n_0+n_1\geqslant n_2+n_3$, $n_0\geqslant n_1$ and that there exists $b\in\min(e_0\cup e_1)\setminus\max(e_2\cup e_3)$. Let $\mathscr{E}=\{A\in\mathscr{A}\mid b\in A\}$ and $\mathscr{D}=\{A\in\mathscr{A}\mid b\notin A\}$. Clearly $e_0,e_1\subseteq\mathscr{E}$ and $e_2,e_3\subseteq\mathscr{D}$. If $n_0+n_1\geqslant|\mathscr{E}|$ then, by (3), there exists an $\mathscr{E}$-spanning (n_0',n_1',e_0,e_1)-doublecycle $\boldsymbol{G}_0$ for $n_1'=\min\{n_1,|\mathscr{E}|/2\}$ and $n_0'=|\mathscr{E}|-n_1'$. Obviously, there exist edges $e_4,e_5\in E(\boldsymbol{G}_0)$ with $e_0\cong_{\boldsymbol{G}_0}e_4$, $e_1\cong_{\boldsymbol{G}_0}e_5$ and such that $n_i'>2$ implies $e_i\neq e_{i+4}$ for $i=0,1$. Then $b\in\min(e_4\cup e_5)\setminus\max(e_2\cup e_3)$, and we set $e_6=\Delta(e_4,b)$, $e_7=\Delta(e_5,b)\subseteq\mathscr{D}$. By the induction hypothesis, there exist an (n_2,n_3,e_2,e_3)-doublecycle $\boldsymbol{G}_1$ and an $(n_0-n_0',n_1-n_1',e_6,e_7)$-doublecycle $\boldsymbol{G}_2$ with $V(\boldsymbol{G}_1)\cap V(\boldsymbol{G}_2)=\emptyset$ and $V(\boldsymbol{G}_1)\cup V(\boldsymbol{G}_2)=\mathscr{D}$. Then $\boldsymbol{G}=(\boldsymbol{G}_0\oplus\boldsymbol{G}_2/\complement(e_4,e_6),\complement(e_5,e_7))$ is an (e_0,e_1,n_0,n_1)-doublecycle with $V(\boldsymbol{G})\cap V(\boldsymbol{G}_1)=\emptyset$, $V(\boldsymbol{G})\cup V(\boldsymbol{G}_1)=\mathscr{A}$. If $n_0+n_1<|\mathscr{E}|$ then we exchange e_0,e_1 and $\mathscr{E}$ with e_2,e_3 and $\mathscr{D}$. Hence the proof of (5) follows.

The proof of (6) is a combination of (4) and (5). $\square$

Since a size of a natural brother of a dense set is uniquely determined by a size of the dense set, we deduce that for every natural number n there exists a unique number $\rho(n)$ such that if $\mathscr{A}$ is a dense set of size n then for every sequence $(\mathscr{B}_0,\mathscr{B}_1,\dots,\mathscr{B}_m)$ of subcubes for which $\mathscr{B}_j$ is a natural brother of a dense set $\mathscr{A}\cup(\bigcup_{i=0}^{j-1}\mathscr{B}_i)$ for all $j=0,1,\dots,m$ (such sequence $(\mathscr{B}_0,\mathscr{B}_1,\dots,\mathscr{B}_m)$ is called a *complementary sequence* of $\mathscr{A}$) with $m\geqslant\rho(n)$ we have $|\mathscr{B}_{\rho(n)}|\geqslant 8$ and either $\rho(n)=0$ or $|\mathscr{B}_{\rho(n)-1}|<8$.

The existence of an $\mathscr{A}$-embedding of a pretty generalized ladder for a dense set $\mathscr{A}$ does not suffice because in the induction step we must connect an embedding of $\boldsymbol{G}_1$ with an embedding of $\boldsymbol{G}_2$, see the beginning of this section. Therefore we define a special dense set that will enable us to proceed with the induction step.

Definition. A triple $(\mathscr{A},A,B)$ is called a *pointed dense set* if $\mathscr{A}$ is a dense set such that $|\mathscr{A}|\geqslant 8$ is even, $|\Delta(A,B)|=2$ and there exist $x\in X$ and a complementary sequence $(\mathscr{B}_0,\mathscr{B}_1,\dots,\mathscr{B}_m)$ of $\mathscr{A}$ with $m\geqslant\rho(|\mathscr{A}|)$ such that if $\mathscr{C}$ is the least subcube containing

A and B (then $\min(\mathscr{C}) = A \cap B$ and $\max(\mathscr{C}) = A \cup B$) then $\Delta(\mathscr{C}, x) \subseteq \mathscr{B}_{\rho(|\mathscr{A}|)}$, and if $\rho(|\mathscr{A}|) > 0$ then $\Delta(\mathscr{C}, \{x, y\})$ and $\mathscr{B}_{\rho(|\mathscr{A}|)-1}$ are neighbouring subcubes for some $y \in \mathrm{dir}(\mathscr{B}_{\rho(|\mathscr{A}|)}) \backslash \Delta(A, B)$.

Let $\boldsymbol{G}$ be a generalized ladder and $(\mathscr{A}, A, B)$ be a pointed dense set of $\boldsymbol{Q}$. We say that an embedding $g : \boldsymbol{G} \to \boldsymbol{Q}$ is a *pointed embedding* onto $(\mathscr{A}, A, B)$ if $\mathrm{Im}(g) = \mathscr{A}$, $g(x_n) = A$ and $g(y_m) = B$.

For any sequence $\mathfrak{A} = (a_0, a_1, a_2, a_3, a_4, a_5)$ of natural numbers such that a_0 and a_1 are odd and a_2, a_3, a_4, and a_5 are even, we define a pretty generalized ladder $\boldsymbol{G}(\mathfrak{A})$ such that: leading paths of $\boldsymbol{G}(\mathfrak{A})$ are $(v_0, v_1, \ldots, v_{a_0+a_4})$ and $(w_0, w_1, \ldots, w_{a_1+a_5})$, rungs of $\boldsymbol{G}(\mathfrak{A})$ are $(u_{0,0}, u_{1,0}, \ldots, u_{a_2,0})$ and $(u_{0,1}, u_{1,1}, \ldots, u_{a_3,1})$ and determining sequences of $\boldsymbol{G}(\mathfrak{A})$ are $(0, a_0)$ and $(0, a_1)$. Thus $u_{00} = v_0$, $u_{a_2,0} = w_0$, $u_{0,1} = v_{a_0}$, $u_{a_3,1} = w_{a_1}$. Clearly, $\boldsymbol{G}(\mathfrak{A})$ is a pretty generalized ladder.

Let $\mathscr{E}_0$ and $\mathscr{E}_1$ be disjoint sets of edges of $\boldsymbol{Q}$. We say that $x \in X$ *separates* $\mathscr{E}_0$ and $\mathscr{E}_1$ if

$$x \in \left(\left(\min\left(\bigcup_{e \in \mathscr{E}_0} e\right)\right) \Big\backslash \left(\max\left(\bigcup_{e \in \mathscr{E}_1} e\right)\right)\right) \cup \left(\left(\min\left(\bigcup_{e \in \mathscr{E}_1} e\right)\right) \Big\backslash \left(\max\left(\bigcup_{e \in \mathscr{E}_0} e\right)\right)\right)$$

Statement 3.3. *For every pretty generalized ladder* $\boldsymbol{G}$ *and for every pointed dense set* $(\mathscr{A}, A, B)$ *with* $|V(\boldsymbol{G})| = |\mathscr{A}|$ *there exists a pointed embedding of* $\boldsymbol{G}$ *onto* $(\mathscr{A}, A, B)$.

Proof. We prove the statement by induction over p. First assume that $p = 0$. Then $\boldsymbol{G}$ consists of two disjoint leading paths $\boldsymbol{P}_0$ of length n and $\boldsymbol{P}_1$ of length m such that n and m are odd. By Statement 1.3(1), there exists a dense set $\mathscr{A}$ with $|\mathscr{A}| = n + m + 2$. It is easy to choose $A, B \in \mathscr{A}$ such that $(\mathscr{A}, A, B)$ is a pointed dense set. Since $(\mathscr{A}, A, B)$ is a pointed dense set, there exist $C, D \subseteq X$ such that $\{A, C\}$ and $\{B, D\}$ are edges of $\boldsymbol{Q}$ contained in $\mathscr{A}$ satisfying the assumptions of Statement 3.2(3). Hence there exists an $\mathscr{A}$-spanning $(n + 1, m + 1, \{A, C\}, \{B, D\})$-doublecycle $\boldsymbol{G}$ and thus there exists a pointed embedding of $\boldsymbol{G}$ onto $(\mathscr{A}, A, B)$. The initial step is proved.

Let $\boldsymbol{G}$ be a pretty generalized ladder with $p > 0$ and let $(\mathscr{A}, A, B)$ be a pointed dense set. Assume that for every pretty generalized ladder $\boldsymbol{G}'$ with the number of rungs less than p and for every pointed dense set $(\mathscr{A}', A', B')$ with $V(\boldsymbol{G}')| = |\mathscr{A}'|$ there exists a pointed embedding f' of $\boldsymbol{G}'$ onto $(\mathscr{A}, A, B)$. Denote $\boldsymbol{G}_1 = _{i_{p-2}-1, j_{p-2}-1}\boldsymbol{G}$ and $\boldsymbol{G}_2 = \boldsymbol{G}_{i_{p-2}, j_{p-2}}$. Clearly, $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are pretty generalized ladders, $\boldsymbol{G}_1$ satisfies the induction hypothesis and $\boldsymbol{G}_2$ is isomorphic to $\boldsymbol{G}(\mathfrak{A})$ for $\mathfrak{A} = (j_{p-1} - j_{p-2}, k_{p-1} - k_{p-2}, \ell_{p-2}, \ell_{p-1}, n - j_{p-1}, m - k_{p-1})$. By Statement 1.3(4) and (6), there exist disjoint dense sets $\mathscr{A}', \mathscr{B}, \mathscr{C}$ with $|\mathscr{A}'| = |V(\boldsymbol{G}_1)|$, $\mathscr{A} = \mathscr{A}' \cup \mathscr{B} \cup \mathscr{C}$ and such that $\mathscr{A}' \cup \mathscr{B}$ is a dense set and $\mathscr{C}$ is contained in its natural brother. Thus we can assume that $\mathscr{B}$ and $\mathscr{C}$ are a-connected for some $a \in X$.

Define a sequence $\mathfrak{B} = (b_0, b_1, b_2, b_3, b_4, b_5)$ such that

$b_0 = \min\{3, j_{p-1} - j_{p-2}\}$;
if $b_0 = 3$ then $b_1 = 1$ else $b_1 = \min\{3, k_{p-1} - k_{p-2}\}$;
if $b_0 + b_1 = 4$ then $b_2 = 2$ else $b_2 = \min\{4, \ell_{p-2}\}$;
if $b_0 + b_1 + b_2 = 6$ then $b_3 = 2$ else $b_3 = \min\{4, \ell_{p-1}\}$;
$b_4 = \min\{2, n - j_{p-1}\}$ and $b_5 = \min\{2, m - k_{p-1}\}$.

Observe that $b_0 + b_1 + b_2 + b_3 \in \{6, 8\}$ and $8 \leqslant b_0 + b_1 + b_2 + b_3 + b_4 + b_5 \leqslant 12$.

By a direct computation generalizing the method from [4], there exist $x, y, z \in X$, $A', B' \subseteq X$, $\mathscr{D}, \mathscr{D}' \subseteq \exp X$, $e_0, e_1, e_2, e_3, e_4, e_5 \in E(\boldsymbol{Q})$ and a $\mathscr{D}'$-spanning embedding h' of $\boldsymbol{G}(\mathfrak{B})$ such that

- $(\mathscr{A}', A', B')$ is a pointed dense set;
- $\mathscr{D}$ is a subcube with $|\mathscr{D}| = 8$,
- $\mathscr{D}'$ is a dense set with $\mathscr{D} \subseteq \mathscr{D}'$, $A, B \in \mathscr{D}'$ and $|\mathscr{D}'| = b_0 + b_1 + b_2 + b_3 + b_4 + b_5$;
- if $\mathscr{B} \cap \mathscr{D}' \neq \emptyset$ then there exists a canonical decomposition $(\mathscr{B}_0, \mathscr{B}_1, \ldots, \mathscr{B}_r)$ of $\mathscr{B}$ and $i_0 \in \{0, 1, \ldots, r\}$ such that $\mathscr{D}' \subseteq \mathscr{B}_{i_0}$ and $\mathscr{B}_{i_0} \setminus \mathscr{D}'$ is a dense set;
- if $\mathscr{C} \cap \mathscr{D}' \neq \emptyset$ then there exists a canonical decomposition $(\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_r)$ of $\mathscr{C}$ and $i_0 \in \{0, 1, \ldots, r\}$ such that $\mathscr{D}' \subseteq \mathscr{C}_{i_0}$ and $\mathscr{C}_{i_0} \setminus \mathscr{D}'$ is a dense set;
- $\Delta(A', x), \Delta(B', x) \in \mathscr{D} \subseteq \mathscr{D}'$;
- $h'(v_0) = \Delta(A', x), h'(w_0) = \Delta(B', x), h'(v_{b_0+b_4}) = A, h'(w_{b_1+b_5}) = B$ and
$$\mathrm{Im}(h') \setminus (\{h(v_i) \mid i > b_0\} \cup \{h(w_i) \mid i > b_1\}) \subseteq \mathscr{D};$$
- e_0 is contained in the h'-image of the path $(v_0, \ldots, v_{b_0})$, e_1 is contained in the h'-image of the path $(w_0, \ldots, w_{b_1})$, e_2 is contained in the h'-image of the path $(u_{0,0}, \ldots, u_{0,b_2})$, e_3 is contained in the h'-image of the path $(u_{1,0}, \ldots, v_1, b_3)$, if the path $(v_{b_0}, \ldots, v_{b_0+b_4})$ is non-empty then e_4 is contained in the h'-image of the path $(v_{b_0}, \ldots, v_{b_0+b_4})$, if the path $(w_{b_1}, \ldots, w_{b_1+b_5})$ is non-empty then e_5 is contained in the h'-image of the path $(w_{b_1}, \ldots, w_{b_1+b_5})$;
- pairs $\{e_0, e_1\}$, $\{e_2, e_3\}$ and $\{e_4, e_5\}$ are parallel, e_0, e_1, e_2, e_3, e_4 and e_5 are pairwise disjoint, z separates $\{e_0, e_1, e_2, e_3\}$ and $\{e_4, e_5\}$, and y separates $\{e_0, e_1\}$ and $\{e_2, e_3\}$.

Observe that either $\mathscr{D}' \cap \mathscr{B} \neq \emptyset$ or $\mathscr{D}' \cap \mathscr{C} \neq \emptyset$ but not both. Denote $n_0 = j_{p-1} - j_{p-2}$, $n_1 = k_{p-1} - k_{p-2}$, $n_2 = \ell_{p-2}$, $n_3 = \ell_{p-1}$, $n_4 = n - j_{p-1}$, and $n_5 = m - k_{p-1}$.

First assume that $\mathscr{D}' \cap \mathscr{B} \neq \emptyset$. Now the proof divides into several cases. Assume $n_4 + n_5 - b_4 - b_5 \neq 0$ and $n_4 - b_4 \geqslant n_5 - b_5$. With no loss of generality, we may assume that $z \notin \min(e_4 \cup e_5)$. Set $\mathscr{E} = \{U \in \mathscr{B} \mid z \in U\}$, $\mathscr{F} = \{U \in \mathscr{B} \mid z \notin U\}$. Then $\mathscr{E}$ and $\mathscr{F}$ are dense sets of $\boldsymbol{Q}$ and $e_4, e_5 \subseteq \mathscr{F}$. Since $\mathscr{B}$ and $\mathscr{C}$ are a-connected, by Statement 1.3(4), we can assume that $\mathscr{E}$ and $\mathscr{C}$ are a-connected for some $a \in X$. Set $n_5' = \min\{n_5 - b_5, |\mathscr{F}|/2\}$ and $n_4' = |\mathscr{F}| - n_5'$. Clearly, there exists $u \in X$ such that if $n_4 > b_4$ then $e_4' = \Delta(e_4, u) \subseteq \mathscr{F} \setminus \mathscr{D}'$, if $n_5 > b_5$ then $e_5' = \Delta(e_5, u) \subseteq \mathscr{F} \setminus \mathscr{D}'$. By Statement 3.1(4), there exists an (n_4', n_5', e_4', e_5')-doublecycle $\boldsymbol{D}_1$ with $V(\boldsymbol{D}_1) \subseteq \mathscr{F}$ such that $\mathscr{F} \setminus V(\boldsymbol{D}_1)$ is a dense set. Assume that $n_4 + n_5 > |\mathscr{F}|$. Obviously, there exist edges e_4'' and e_5''

with $e_6 = \Delta(e_4'', b)$, $e_7 = \Delta(e_5'', b) \subseteq \mathscr{B}_r \cap \mathscr{E}$ and such that if $n_4 > b_4$ then $e_4'' \in E(\boldsymbol{D}_1)$ and if $n_4 > b_4 - 2$ then $e_4'' \neq e_4'$, if $n_5 > b_5$ then $e_5'' \in E(\boldsymbol{D}_1)$ and if $n_5 > b_5 - 2$ then $e_5'' \neq e_5'$. From Statement 1.3(4) it follows that $\mathscr{B}_r \cup \mathscr{C}$ is a dense set. Since $\mathscr{E}$ and $\mathscr{C}$ are a-connected for some $a \in X$, by Statement 3.2(5), we obtain that there exists an $(n_4 - n_4' - b_4, n_5 - n_5' - b_5, e_6, e_7)$-doublecycle $\boldsymbol{D}_2$ with $V(\boldsymbol{D}_2) \subseteq \mathscr{E} \cup \mathscr{C}$ and such that $\mathscr{E} \backslash V(\boldsymbol{D}_2)$ is a dense set, if $n_i > b_i$ then there exists $x_i \in X$ with $\Delta(e_i, x_i) \subseteq \mathscr{E} \backslash V(\boldsymbol{D}_2)$ for all $i = 0, 1, 2, 3$ and if $\mathscr{C} \backslash V(\boldsymbol{D}_2)$ is a dense set then $\mathscr{E} \backslash V(\boldsymbol{D}_2)$ and $\mathscr{C} \backslash V(\boldsymbol{D}_2)$ are a-connected. Set $\boldsymbol{D}_0 = (\boldsymbol{D}_1 \oplus \boldsymbol{D}_2 \backslash \complement(e_4'', e_6), \complement(e_5'', e_7))$. Then

(A) $\boldsymbol{D}_0$ is an $(n_4 - b_4, n_5 - b_5, e_4', e_5')$-doublecycle with $V(\boldsymbol{D}_0) \subseteq \mathscr{B} \cup \mathscr{C}$ and such that $\mathscr{B} \backslash V(\boldsymbol{D}_0)$ is a dense set, if $n_i > b_i$ then there exists $x_i \in X$ with $\Delta(e_i, x_i) \subseteq \mathscr{B} \backslash V(\boldsymbol{D}_0)$ for all $i = 0, 1, 2, 3$ and if $\mathscr{C} \backslash V(\boldsymbol{D}_0) \neq \emptyset$ then $\mathscr{C} \backslash V(\boldsymbol{D}_0)$ is a dense set and $\mathscr{B} \backslash V(\boldsymbol{D}_0)$ and $\mathscr{C} \backslash V(\boldsymbol{D}_0)$ are a-connected.

If $n_4 + n_5 - b_4 - b_5 \leqslant |\mathscr{F}|$ then set $\boldsymbol{D}_0 = \boldsymbol{D}_1$ and $\boldsymbol{D}_0$ again satisfies (A). If $\mathscr{D}' \cap \mathscr{B} \neq \emptyset$ and $n_4 + n_5 - b_4 - b_5 = 0$ then set $\boldsymbol{D}_0$ is the empty graph and $\boldsymbol{D}_0$ again satisfies (A).

Now we apply Statement 3.2(6) for $\Delta(e_i, x_i)$ and $n_i - b_i$, $i = 0, 1, 2, 3$ to obtain an $(n_0 - b_0, n_1 - b_1, \Delta(e_0, x_0), \Delta(e_1, x_1))$-doublecycle $\boldsymbol{D}_3$ and an $(n_2 - b_2, n_3 - b_3, \Delta(e_2, x_2), \Delta(e_3, x_4))$-doublecycle $\boldsymbol{D}_4$ such that $\boldsymbol{D}_0$, $\boldsymbol{D}_3$ and $\boldsymbol{D}_4$ are pairwise disjoint, $V(\boldsymbol{D}_0) \cup V(\boldsymbol{D}_3) \cup V(\boldsymbol{D}_4) = (\mathscr{B} \cup \mathscr{C}) \backslash \mathscr{D}'$. By merging h' with $\boldsymbol{D}_0$, $\boldsymbol{D}_3$ and $\boldsymbol{D}_4$, we deduce that there exists a $\mathscr{B} \cup \mathscr{C}$-spanning embedding h of $\boldsymbol{G}_2$ with $h(v_0) = h(x_{j_{p-2}}) = \Delta(A', x), h(w_0) = h(y_{k_{p-2}}) = \Delta(B', x), h(v_{n-j_{p-2}}) = h(x_n) = A, h(w_{m-k_{p-2}}) = h(y_m) = B$. By the induction hypothesis, there exists a pointed embedding g of $\boldsymbol{G}_1$ onto $(\mathscr{A}', A', B')$. Thus the union f of g and h is a pointed embedding of $\boldsymbol{G}$ onto $(\mathscr{A}, A, B)$.

If $\mathscr{C} \cap \mathscr{D}' \neq \emptyset$ then the proof is analogous, we exchange $\mathscr{B}$ and $\mathscr{C}$. The proof is complete. $\square$

Since any cellular ladder (see [4]) is a pretty generalized ladder, Theorem 3.4 that is a consequence of the above result generalizes Theorem 5.1 of [4].

As a consequence we obtain the main theorem of this section.

Theorem 3.4. *Every pretty generalized ladder $\boldsymbol{G}$ has an optimal embedding.*

Corollary 2.2 and Theorem 3.4 are similar, yet the methods of their proofs are distinct. We do not know whether both results can be demonstrated by a unique method. In particular, it appears that the method of the proof of Theorem 2.1 — and originally the method from [2] — cannot be used for generalized ladders with odd rungs (because in this case, there exists a ladder with odd rungs without optimal embedding) and the method of the proof of Statement 3.3 — originally, in [4] — cannot be used for cyclic generalized ladders (because the vertices $f(x_0)$ and $f(y_0)$ may not be close to A and B). Therefore, some questions remain open, for instance, the characterization of balanced bipartite cyclic generalized ladders with odd rungs that have an optimal embedding into a hypercube.

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